Lifshits tails caused by anisotropic decay: the emergence of a quantum-classical regime

Werner Kirsch and Simone Warzel

ABSTRACT. We investigate Lifshits-tail behaviour of the integrated density of states for a wide class of Schrödinger operators with positive random potentials. The setting includes alloy-type and Poissonian random potentials. The considered (single-site) impurity potentials $f : \mathbb{R}^d \to [0,\infty[$ decay at infinity in an anisotropic way, for example, $f(x_1,x_2) \sim (|x_1|^{\alpha_1}+|x_2|^{\alpha_2})^{-1}$ as $|(x_1,x_2)| \to \infty$. As is expected from the isotropic situation, there is a so-called quantum regime with Lifshits exponent $d/2$ if both α_1 and α_2 are big enough, and there is a so-called classical regime with Lifshits exponent depending on α_1 and α_2 if both are small. In addition to this we find two new regimes where the Lifshits exponent exhibits a mixture of quantum and classical behaviour. Moreover, the transition lines between these regimes depend in a nontrivial way on α_1 and α_2 simultaneously.

Dedicated to the memory of G. A. Mezincescu (1943 – 2001).

CONTENTS

Key words and phrases. Random Schrödinger operators, Integrated density of states, Lifshits tails.

1. Introduction

The integrated density of states $N : \mathbb{R} \to [0, \infty[$ is an important basic quantity in the theory of disordered electronic systems [**Kir89, CL90, Lan91, PF92, Sto01, LMW03, Ves03**]. Roughly speaking, $N(E)$ describes the number of energy levels below a given energy E per unit volume (see (15) below for a precise definition). A characteristic feature of disordered systems is the behaviour of N near band edges. It was first studied by Lifshits [**Lif63**]. He gave convincing physical arguments that the polynomial decrease

$$\log N(E) \sim \log\,(E - E_0)^{\frac{d}{2}} \qquad \text{as} \quad E \downarrow E_0 \tag{1}$$

known as van-Hove singularity (see [**KS87**] for a rigorous proof) near a band edge E_0 of an ideal periodic system in d space dimensions is replaced by an exponential decrease in a disordered system. In his honour, this decrease is known as Lifshits singularity or Lifshits tail and typically given by

$$\log N(E) \sim \log e^{-C\,(E-E_0)^{-\eta}} \qquad \text{as} \quad E \downarrow E_0 \tag{2}$$

where $\eta > 0$ is called the Lifshits exponent and $C > 0$ is some constant.

The first rigorous proof [**DV75**] (see [**Nak77**]) of Lifshits tails (in the sense that (2) holds) concerns the bottom E_0 of the energy spectrum of a continuum model involving a Poissonian random potential

$$V_\omega(x) := \sum_j f(x - \xi_{\omega,j}), \tag{3}$$

where $\xi_{\omega,j} \in \mathbb{R}^d$ are Poisson distributed points and $f : \mathbb{R}^d \to [0, \infty[$ is a non-negative impurity potential. Donsker and Varadhan [**DV75**] particularly showed that the Lifshits exponent is universally given by $\eta = d/2$ in case

$$0 \le f(x) \;\le\; f_0\,(1 + |x|)^{-\alpha} \quad \text{with some } \alpha > d + 2 \text{ and some } f_0 > 0. \tag{4}$$

It was Pastur [**Pas77**] who proved that the Lifshits exponent changes to $\eta = d/(\alpha - d)$ if

$$f_u\,(1 + |x|)^{-\alpha} \le f(x) \le f_0\,(1 + |x|)^{-\alpha} \qquad \begin{array}{c}\text{with some } d < \alpha < d + 2 \\ \text{and some } f_u,\, f_0 > 0.\end{array} \tag{5}$$

This change from a universal Lifshits exponent to a non-universal one, which depends on the decay exponent α of f, may be heuristically explained in terms of a competition of the kinetic and the potential energy of the underlying one-particle Schrödinger operator. In the first case ($\eta = d/2$) the quantum mechanical kinetic energy has a crucial influence on the (first order) asymptotics of N. The Lifshits tail is then said to have a *quantum* character. In the other case it is said to have a *classical* character since then the (classical) potential energy determines the asymptotics of N. For details, see for example [**Lan91, PF92, LW04**].

Analogous results have been obtained for other random potentials. For example, in case of an alloy-type random potential

$$V_\omega(x) := \sum_{j \in \mathbb{Z}^d} q_{\omega,j}\, f(x - j) \tag{6}$$

which is given in terms of independent identically distributed random variables $q_{\omega,j}$ and an impurity potential $f : \mathbb{R}^d \to [0, \infty[$, the Lifshits tails at the lowest band edge E_0 have been investigated by [**KM83a, KS86, Mez87**]. Similarly to the Poissonian case the authors of

[**KS86, Mez87**] consider f as in (4) and (5) and detect a quantum and a classical regime for which the Lifshits exponent equals

$$\eta = \left\{ \begin{array}{lll} \frac{d}{2} & \text{in case } (4): & d+2<\alpha \\ \frac{d}{\alpha-d} & \text{in case } (5): & d<\alpha<d+2 \end{array} \right\} = \max\left\{\frac{d}{2}, \frac{d/\alpha}{1-d/\alpha}\right\} \tag{7}$$

In fact they do not obtain the asymptotics (2) on a logarithmic scale but only double-logarithmic asymptotics (confer (16) below). (See also [**Sto99**] for an alternative proof of this double-logarithmic asymptotics in case of alloy-type and Poissonian random potentials.)

Our main point is to generalise these results on the Lifshits exponent to impurity potentials f that decay in an *anisotropic* way at infinity (confer (8) below). In addition we are able to handle a wide class of random potentials given in terms of random Borel measures which include among further interesting examples both the case of alloy-type potentials and Poisson potential. Thus the *same* proof works for these two most important cases.

In our opinion it is interesting to explore the transition between quantum and classical Lifshits behaviour in such models from both a mathematical and a physical point of view. The interesting cases are those for which f decays fast enough in some directions to ensure a quantum character while it decays slowly in the other direction so that the expected character there is the classical one. In the following we give a complete picture of the classical and the quantum regime of the integrated density of states as well as of the emerging mixed *quantum-classical* regime. We found it remarkable that the borderline between the quantum and classical behaviour caused by the decay of f in a certain direction is not determined by the corresponding decay exponent of these directions alone, but depends also in a nontrivial way on the decay in the other directions.

A second motivation for this paper came from investigations of the Lifshits tails in a constant magnetic field in three space dimensions [**War01, HKW03, LW04**]. In contrast to the two-dimensional situation [**BHKL95, Erd98, HLW99, HLW00, Erd01, War01**], the magnetic field introduces an anisotropy in $\mathbb{R}^3$, such that it is quite natural to look at f which are anisotropic as well. In fact, in the three-dimensional magnetic case a quantum-classical regime has already been shown to occur for certain f with isotropic decay [**War01, LW04**]. The present paper will contribute to a better understanding of these results.

The results mentioned above as well as the results in this paper concern Lifshits tails at the bottom of the spectrum. In accordance with Lifshits' heuristics, the integrated density of states should behave in a similar way at other edges of the spectrum. Such internal Lifshits tails were proven in [**Mez86, Sim87, Mez93, Klo99, KW02, Klo02**].

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2. Basic quantities and main result

2.1. Random potentials. We consider random potentials

$$V : \Omega \times \mathbb{R}^d \to [0,\infty[, \quad (\omega, x) \mapsto V_\omega(x) := \int_{\mathbb{R}^d} f(x-y)\,\mu_\omega(dy), \tag{8}$$

which are given in terms of a random Borel measure $\mu : \Omega \to \mathcal{M}(\mathbb{R}^d)$, $\omega \mapsto \mu_\omega$, and an impurity potential $f : \mathbb{R}^d \to [0,\infty[$. We recall from [**Kal83, SKM87, DVJ88**] that a random Borel measure is a measurable mapping from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the set of Borel measures $\big(\mathcal{M}(\mathbb{R}^d), \mathcal{B}(\mathcal{M})\big)$, that is, the set of positive, locally-finite measures

on $\mathbb{R}^d$. Here $\mathcal{B}(\mathcal{M})$ denotes the Borel σ-algebra of $\mathcal{M}(\mathbb{R}^d)$, that is, the smallest σ-algebra rendering the mappings $\mathcal{M}(\mathbb{R}^d) \ni \nu \mapsto \nu(\Lambda)$ measurable for all bounded Borel sets $\Lambda \in \mathcal{B}(\mathbb{R}^d)$.

The following assumptions on μ are supposed to be valid throughout the paper.

Assumption 2.1. The random Borel measure $\mu : \Omega \to \mathcal{M}(\mathbb{R}^d)$, $\omega \mapsto \mu_\omega$ is defined on some complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We suppose that:

(i) μ is $\mathbb{Z}^d$-stationary.

(ii) there exists a partition of $\mathbb{R}^d = \bigcup_{j\in\mathbb{Z}^d} \Lambda_j$ into disjoint unit cubes $\Lambda_j = \Lambda_0 + j$ centred at the sites of the lattice $\mathbb{Z}^d$ such that the random variables $\big(\mu(\Lambda^{(j)})\big)_{j\in J}$ are stochastically independent for any finite collection $J \subset \mathbb{Z}^d$ of Borel sets $\Lambda^{(j)} \subset \Lambda_j$.

(iii) the intensity measure $\overline{\mu} : \mathcal{B}(\mathbb{R}^d) \to [0,\infty[$, which is given by

$$\overline{\mu}(\Lambda) := \mathbb{E}\big[\mu(\Lambda)\big] \tag{9}$$

in terms of the probabilistic expectation $\mathbb{E}[\cdot] := \int_\Omega (\cdot)\, \mathbb{P}(d\omega)$, is a Borel measure which does not vanish identically $\overline{\mu} \neq 0$.

(iv) there is some constant $\kappa > 0$ such that $\mathbb{P}\left\{\omega \in \Omega : \mu_\omega(\Lambda_0) \in [0,\varepsilon[\right\} \geq \varepsilon^\kappa$ for small enough $\varepsilon > 0$.

Remark 2.2. Assumption 2.1(i) implies that the intensity measure $\overline{\mu}$ is $\mathbb{Z}^d$-periodic. Assumption 2.1(iii) is thus equivalent to the existence of the first moment $\mathbb{E}\left[\mu(\Lambda_0)\right] < \infty$ of the random variable $\mu(\Lambda_0) : \omega \mapsto \mu_\omega(\Lambda_0)$. Moreover, we emphasis that the unit cubes (Λ_j) introduced in Assumption 2.1(ii) are neither open nor closed.

We recall from [**Kal83, SKM87, DVJ88**] that $\mathbb{Z}^d$-stationarity of μ requires the group $(T_j)_{j\in\mathbb{Z}^d}$ of lattice translations, which is defined on $\mathcal{M}(\mathbb{R}^d)$ by $(T_j\nu)(\Lambda) := \nu(\Lambda + j)$ for all $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ and all $j \in \mathbb{Z}^d$, to be probability preserving in the sense that

$$\mathcal{P}\left\{T_j M\right\} = \mathcal{P}\left\{M\right\} \tag{10}$$

for all $M \in \mathcal{B}(\mathcal{M})$ and all $j \in \mathbb{Z}^d$. Here we have introduced the notation $\mathcal{P}\left\{M\right\} := \mathbb{P}\left\{\omega \in \Omega : \mu_\omega \in M\right\}$ for the induced probability measure on $\big(\mathcal{M}(\mathbb{R}^d), \mathcal{B}(\mathcal{M})\big)$. To ensure the ($\mathbb{Z}^d$-)ergodicity of the random potential V, it is useful to know that under the assumptions made above, (T_j) is a group of mixing (hence ergodic) transformations on the probability space $\big(\mathcal{M}(\mathbb{R}^d), \mathcal{B}(\mathcal{M}), \mathcal{P}\big)$.

Lemma 2.3. *Assumption 2.1(i) and 2.1(ii) imply that μ is mixing in the sense that*

$$\lim_{|j|\to\infty} \mathcal{P}\left\{T_j M \cap M'\right\} = \mathcal{P}\left\{M\right\} \mathcal{P}\left\{M'\right\} \tag{11}$$

for all $M, M' \in \mathcal{B}(\mathcal{M})$.

PROOF. See Appendix A. □

The considered impurity potentials $f : \mathbb{R}^d \to [0,\infty[$ comprise a large class of functions with anisotropic decay. More precisely, we decompose the configuration space $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_m}$ into $m \in \mathbb{N}$ subspaces with dimensions $d_1, \dots, d_m \in \mathbb{N}$. Accordingly, we will write $x = (x_1, \dots, x_m) \in \mathbb{R}^d$, where $x_k \in \mathbb{R}^{d_k}$ and $k \in \{1, \dots, m\}$. Denoting by $|x_k| := \max_{i\in\{1,\dots,d_k\}} |(x_k)_i|$ the maximum norm on $\mathbb{R}^{d_k}$, our precise assumptions on f are as follows.

Assumption 2.4. The impurity potential $f : \mathbb{R}^d \to [0, \infty[$ is positive, strictly positive on some non-empty open set and satisfies:

(i) the Birman-Solomyak condition $\sum_{j\in\mathbb{Z}^d} \big(\int_{\Lambda_0} |f(x-j)|^p dx\big)^{1/p} < \infty$ with $p = 2$ if $d \in \{1, 2, 3\}$ and $p > d/2$ if $d \geq 4$.

(ii) there exist constants $\alpha_1, \ldots, \alpha_m \in [0, \infty]$ and $0 < f_u, f_0 < \infty$ such that

$$\frac{f_u}{\sum_{k=1}^m |x_k|^{\alpha_k}} \leq \int_{\Lambda_0} f(y-x)\, dy, \qquad f(x) \leq \frac{f_0}{\sum_{k=1}^m |x_k|^{\alpha_k}} \tag{12}$$

for all $x = (x_1, \ldots, x_m) \in \mathbb{R}^d$ with large enough values of their maximum norm $|x| = \max\{|x_1|, \ldots, |x_m|\}$.

Remark 2.5. In order to simultaneously treat the case $\alpha_k = \infty$ for some (or all) $k \in \{1, \ldots, m\}$, we adopt the conventions $|x_k|^\infty := \infty$ for $|x_k| > 0$ and $1/\infty := 0$. An example for such a situation is given by f with compact support in the x_k-direction.

2.2. Examples. The setting in Subsection 2.1 covers a huge class of random potentials which are widely encountered in the literature on random Schrödinger operators [**Kir89, CL90, PF92, Sto01**]. In this Subsection we list prominent examples, some of which have already been (informally) introduced in the Introduction.

From the physical point of view, it natural to consider integer-valued random Borel measures $\nu = \sum_j k_j \delta_{x_j}$, also known as point processes [**DVJ88**]. Here each k_j is an integer-valued random variable and the distinct points (x_j) indexing the atoms, equivalently the Dirac measure δ, form a countable (random) set with at most finitely many x_j in any bounded Borel set. In fact, interpreting (x_j) as the (random) positions of impurities in a disordered solid justifies the name 'impurity potential' for f in (8).

Two examples of point processes satisfying Assumptions 2.1(i)–2.1(iii) are:

(P) the *generalised Poisson measure* $\nu = \sum_j \delta_{\xi_j}$ with some non-zero $\mathbb{Z}^d$-periodic Borel intensity measure $\overline{\nu}$. The Poisson measure is uniquely characterised by requiring that the random variables $\nu(\Lambda^{(1)}), \ldots, \nu(\Lambda^{(n)})$ are stochastically independent for any collection of disjoint Borel sets $\Lambda^{(1)}, \ldots \Lambda^{(n)} \in \mathcal{B}(\mathbb{R}^d)$ and that each $\nu(\Lambda)$ is distributed according to Poisson's law

$$\mathbb{P}\big\{\omega \in \Omega : \nu_\omega(\Lambda) = k\big\} = \frac{\big(\overline{\nu}(\Lambda)\big)^k}{k!} \exp\big[-\overline{\nu}(\Lambda)\big], \quad k \in \mathbb{N}_0 \tag{13}$$

for any bounded $\Lambda \in \mathcal{B}(\mathbb{R}^d)$. The case $\overline{\nu}(\Lambda) = \varrho|\Lambda|$ corresponds to the usual *Poisson process* with parameter $\varrho > 0$.

(D) the *displacement measure* $\nu = \sum_{j\in\mathbb{Z}^d} \delta_{j+d_j}$. Here the random variables $d_j \in \Lambda_0$ are independent and identically distributed over the unit cube. The case $d_j = 0$ corresponds to the (non-random) *periodic point measure* $\nu = \sum_{j\in\mathbb{Z}^d} \delta_j$.

Any (generalised) Poisson measure (P) also satisfies Assumption 2.1(iv). It gives rise to the (generalized) Poissonian random potential (3). Unfortunately, Assumption 2.1(iv) is never satisfied for any displacement measure (D). However, a corresponding compound point process $\nu = \sum_{j\in\mathbb{Z}^d} q_j \delta_{x_j}$ will satisfy Assumption 2.1(iv) under suitable conditions on the random variables (q_j). In order to satisfy Assumption 2.1(iii), we take $(q_j)_{j\in\mathbb{Z}^d}$ independent and identically distributed, positive random variables with $0 < \mathbb{E}[q_0] < \infty$.

Two examples of such compound point processes, for which Assumptions 2.1(i)–2.1(iv) hold, are:

(P') the *compound (generalised) Poisson measure* $\nu = \sum_j q_j \delta_{\xi_j}$ with (ξ_j) as in (P).

(D') the *compound displacement measure* $\nu = \sum_{j\in\mathbb{Z}^d} q_j \delta_{j+d_j}$ with d_j as in (D). Assumption 2.1(iv) requires $\mathbb{P}\big\{\omega \in \Omega : \, q_{\omega,0} \in [0,\varepsilon[\big\} \geq \varepsilon^\kappa$ for small enough $\varepsilon > 0$ and some $\kappa > 0$. The case $d_j = 0$ gives the *alloy-type measure* $\nu = \sum_{j\in\mathbb{Z}^d} q_j \delta_j$ associated with the alloy-type random potential (6).

Remark 2.6. We note that in case (P') there are no further requirements on (q_j). Moreover, our results in Subsection 2.4 below also apply to alloy-type random potentials (6) with bounded below random variables (q_j), not only positive ones. This follows from the fact that one may add $x \mapsto \sum_{j\in\mathbb{Z}^d} q_{\min} f(x-j)$ to the periodic background potential $U_{\rm per}$ (confer (14) and Assumption 2.7 below).

2.3. Random Schödinger operators and their integrated density of states. For any of the above defined random potentials V, we study the corresponding random Schrödinger operator, which is informally given by the second order differential operator

$$H(V_\omega) := -\Delta + U_{\rm per} + V_\omega \tag{14}$$

on the Hilbert space $\mathrm{L}^2(\mathbb{R}^d)$ of complex-valued, square-integrable functions on $\mathbb{R}^d$. Thereby the periodic background potential $U_{\rm per}$ (acting in (14) as a multiplication operator) is required to satisfy the following

Assumption 2.7. The background potential $U_{\rm per} : \mathbb{R}^d \to \mathbb{R}$ is $\mathbb{Z}^d$-periodic and $U_{\rm per} \in \mathrm{L}^p_{\rm loc}\left(\mathbb{R}^d\right)$ for some $p > d$.

Assumptions 2.1 and 2.4 particularly imply [**CL90**, Cor. V.3.4] that $V_\omega \in \mathrm{L}^p_{\rm loc}(\mathbb{R}^d)$ for $\mathbb{P}$-almost all $\omega \in \Omega$ with the same p as in Assumption 2.4(ii). Together with Assumption 2.7 this ensures [**KM83b**] that $H(V_\omega)$ is essentially self-adjoint on the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ of complex-valued, arbitrarily often differentiable functions with compact support for $\mathbb{P}$-almost all $\omega \in \Omega$. Since V is $\mathbb{Z}^d$-ergodic (confer Lemma 2.3), the spectrum of $H(V_\omega)$ coincides with a non-random set for $\mathbb{P}$-almost all $\omega \in \Omega$ [**KM82**, Thm. 1].

For any d-dimensional open cuboid $\Lambda \subset \mathbb{R}^d$, the restriction of (14) to $\mathcal{C}_c^\infty(\Lambda)$ defines a self-adjoint operator $H_\Lambda^D(V_\omega)$ on $L^2(\Lambda)$, which corresponds to taking Dirichlet boundary conditions [**RS78**]. It is bounded below and has purely discrete spectrum with eigenvalues $\lambda_0(H_\Lambda^D(V_\omega) < \lambda_1(H_\Lambda^D(V_\omega) \leq \lambda_2(H_\Lambda^D(V_\omega) \leq \dots$ ordered by magnitude and repeated according to their multiplicity. Our main quantity of interest, the integrated density of states, is then defined as the infinite-volume limit

$$N(E) := \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \,\#\Big\{ n \in \mathbb{N}_0 \,:\, \lambda_n\left(H_\Lambda^D(V_\omega)\right) < E \Big\} \tag{15}$$

More precisely, thanks to the $\mathbb{Z}^d$-ergodicity of the random potential there is a set $\Omega_0 \in \mathcal{A}$ of full probability, $\mathbb{P}(\Omega_0) = 1$, and a non-random unbounded distribution function $N : \mathbb{R} \to [0,\infty[$ such that (15) holds for all $\omega \in \Omega_0$ and all continuity points $E \in \mathbb{R}$ of N. The set of growth points of N coincides with the almost-sure spectrum of $H(V_\omega)$, confer [**Kir89, CL90, PF92**].

2.4. Lifshits tails. The main result of the present paper generalises the result (7) of [**KS86, Mez87**] on the Lifshits exponent for alloy-type random potentials with isotropically decaying impurity potential f to the case of anisotropic decay and more general random potentials (8). We note that isotropic decay corresponds to taking $m = 1$ in Assumption 2.4 or, what is the same, $\alpha := \alpha_k$ for all $k \in \{1, \dots, m\}$.

Theorem 2.8. *Let $H(V_\omega)$ be a random Schrödinger operator (14) with random potential (8) satisfying Assumptions 2.1 and 2.4, and a periodic background potential satisfying Assumption 2.7. Then its integrated density of states N drops down to zero exponentially near $E_0 := \inf \operatorname{spec} H(0)$ with Lifshits exponent given by*

$$\eta := \lim_{E \downarrow E_0} \frac{\log|\log N(E)|}{|\log(E - E_0)|} = \sum_{k=1}^{m} \max\left\{\frac{d_k}{2}, \frac{\gamma_k}{1-\gamma}\right\}, \tag{16}$$

where $\gamma_k := d_k/\alpha_k$ and $\gamma := \sum_{k=1}^{m} \gamma_k$.

Remarks 2.9. (i) As a by-product, it turns out that the infimum of the almost-sure spectrum of $H(V_\omega)$ coincides with that of $H(0) = -\Delta + U_{\mathrm{per}}$.

(ii) Thanks to the convention $0 = d_k/\infty \, (= \gamma_k)$, Theorem 2.8 remains valid if $\alpha_k = \infty$ for some (or all) $k \in \{1, \dots, m\}$, confer Remark 2.5.

(iii) Assumption 2.7 on the local singularities of U_{per} is slightly more restrictive than the one in [**KS86, Mez87**]. It is tailored to ensure certain regularity properties of the ground-state eigenfunction of $H(0)$. As can be inferred from Subsection 3.1 below, we may relax Assumption 2.7 and require only $p > d/2$ (as in [**KS86, Mez87**]) in the interior of the unit cube and thus allow for Coulomb singularities there.

(iv) Even in the isotropic situation $m = 1$ Assumption 2.4 covers slightly more impurity potentials than in [**KS86, Mez87**], since we allow f to have zeros at arbitrary large distance from the origin.

(v) An inspection of the proof below shows that we prove a slightly better estimate than the double logarithmic asymptotics given in (16). In particular, if the measure μ_ω has an atom at zero, more exactly if $\mathbb{P}\{\omega \in \Omega : \mu_\omega(\Lambda_0) = 0\} > 0$, then we actually prove

$$-C \, (E - E_0)^\eta \le \log N(E) \le -C' \, (E - E_0)^\eta \tag{17}$$

for small $E - E_0$. This is not quite the logarithmic behaviour (2) of N since the constants $C > 0$ and $C' > 0$ do not agree. Note that μ_ω has an atom at zero for the any generalized Poisson measure (P) as well as for a compound displacement measure (D') if $\mathbb{P}\{\omega \in \Omega : q_{\omega,0}(\omega) = 0\} > 0$.

For an illustration and interpretation of Theorem 2.8 we consider the special case $m = 2$. The right-hand side of (16) then suggests to distinguish the following three cases:

Quantum regime:	$\dfrac{d_1}{2} \ge \dfrac{\gamma_1}{1-\gamma}$	and	$\dfrac{d_2}{2} \ge \dfrac{\gamma_2}{1-\gamma}$.	(qm)
Quantum-classical regime:	$\dfrac{d_1}{2} \ge \dfrac{\gamma_1}{1-\gamma}$	and	$\dfrac{d_2}{2} < \dfrac{\gamma_2}{1-\gamma}$	(qm/cl)
or:	$\dfrac{d_1}{2} < \dfrac{\gamma_1}{1-\gamma}$	and	$\dfrac{d_2}{2} \ge \dfrac{\gamma_2}{1-\gamma}$	(cl/qm)
Classical regime:	$\dfrac{d_1}{2} < \dfrac{\gamma_1}{1-\gamma}$	and	$\dfrac{d_2}{2} < \dfrac{\gamma_2}{1-\gamma}$	(cl)

In comparison to the result (7) for $m = 1$ the main finding of this paper is the emergence of a regime corresponding to mixed quantum and classical character of the Lifshits tail. A remarkable fact about the Lifshits exponent (16) is that the directions $k \in \{1, 2\}$ related to the anisotropy do not show up separately as one might expect naively. In particular, the transition from a quantum to a classical regime for the x_k-direction does not occur if $d_k/2 = \gamma_k/(1-\gamma_k)$, but rather if $d_k/2 = \gamma_k/(1-\gamma)$. This intriguing intertwining of directions through γ may be interpreted in terms of the marginal impurity potentials $f^{(1)}$ and $f^{(2)}$ defined in (24) and (25) below. In fact, when writing $\gamma_2/(1-\gamma) = d_2/\left(\alpha_2(1-\gamma_1) - d_2\right)$ and identifying $\alpha_2(1-\gamma_1)$ as the decay exponent of $f^{(2)}$ by Lemma 3.4 below, it is clear that $f^{(2)}$ serves as an effective potential for the x_2-direction as far as the quantum-classical transition is concerned. In analogy, $f^{(1)}$ serves as the effective potential for the x_1-direction. Heuristic arguments for the importance of the marginal potentials in the presence of an anisotropy can be found in [**LW04**].

3. Basic inequalities and auxiliary results

In order to keep our notation as transparent as possible, we will additionally suppose that

$$E_0 = 0 \qquad \text{and} \qquad m = 2 \tag{18}$$

throughout the subsequent proof of Theorem 2.8. In fact, the first assumption can always be achieved by adding a constant to $H(0)$.

The strategy of the proof is roughly the same as in [**KS86, Mez87**], which in turn is based on [**KM83a, Sim85**]. We use bounds on the integrated density of states N and subsequently employ the Rayleigh-Ritz principle and Temple's inequality [**RS78**] to estimate the occurring ground-state energies from above and below. The basic idea to construct the bounds on N is to partition the configuration space $\mathbb{R}^d$ into congruent domains and employ some bracketing technique for $H(V_\omega)$. The most straightforward of these techniques is Dirichlet or Neumann bracketing. However, to apply Temple's inequality to the arising Neumann ground-state energy, the authors of [**KS86**] required that U_{per} is reflection invariant. To get rid of this additional assumption, Mezincescu [**Mez87**] suggested an alternative upper bound on N which is based on a bracketing technique corresponding to certain Robin (mixed) boundary conditions. In his honour, we will refer to these particular Robin boundary conditions as Mezincescu boundary conditions.

3.1. Mezincescu boundary conditions and basic inequalities.

Assumption 2.7 on U_{per} implies [**Sim82**, Thm. C.2.4] that there is a continuously differentiable representative $\psi : \mathbb{R}^d \to]0, \infty[$ of the strictly positive ground-state eigenfunction of $H(0) = -\Delta + U_{\mathrm{per}}$, which is L^2-normalised on the unit cube Λ_0,

$$\int_{\Lambda_0} \psi(x)^2 dx = 1. \tag{19}$$

The function ψ is $\mathbb{Z}^d$-periodic, bounded from below by a strictly positive constant and obeys $H(0)\psi = E_0\psi = 0$.

Subsequently, we denote by $\Lambda \subset \mathbb{R}^d$ a d-dimensional, open cuboid which is compatible with the lattice $\mathbb{Z}^d$, that is, we suppose that it coincides with the interior of the union of $\mathbb{Z}^d$-translates of the closed unit cube. On the boundary $\partial\Lambda$ of Λ we define $\chi : \partial\Lambda \to \mathbb{R}$ as the negative of the outer normal derivative of $\log \psi$,

$$\chi(x) := -\frac{1}{\psi(x)} \, (n \cdot \nabla) \, \psi(x), \quad x \in \partial\Lambda. \tag{20}$$

Since $\chi \in L^\infty(\partial\Lambda)$ is bounded, the sesquilinear form

$$(\varphi_1, \varphi_2) \mapsto \int_\Lambda \overline{\nabla\varphi_1(x)} \cdot \nabla\varphi_2(x)\, dx + \int_{\partial\Lambda} \chi(x)\, \overline{\varphi_1(x)}\varphi_2(x)\, dx, \tag{21}$$

with domain $\varphi_1, \varphi_2 \in W^{1,2}(\Lambda) := \big\{\varphi \in L^2(\Lambda) : \nabla_j\varphi \in L^2(\Lambda) \text{ for all } j \in \{1, \ldots, d\}\big\}$, is symmetric, closed and lower bounded, and thus uniquely defines a self-adjoint operator $-\Delta_\Lambda^\chi =: H_\Lambda^\chi(0) - U_{\rm per}$ on $L^2(\Lambda)$. In fact, the condition $\chi \in L^\infty(\partial\Lambda)$ guarantees that boundary term in (21) is form-bounded with bound zero relative to the first term, which is just the quadratic form corresponding to the (negative) Neumann Laplacian. Consequently [**RS78**, Thm. XIII.68], both the Robin Laplacian $-\Delta_\Lambda^\chi$ as well as $H_\Lambda^\chi(V_\omega) := -\Delta_\Lambda^\chi + U_{\rm per} + V_\omega$, defined as a form sum on $W^{1,2}(\Lambda) \subset L^2(\Lambda)$, have compact resolvents. Since $H_\Lambda^\chi(V_\omega)$ generates a positivity preserving semigroup, its ground-state is simple and comes with a strictly positive eigenfunction [**RS78**, Thm. XIII.43].

Remarks 3.1. (i) In the boundary term in (21) we took the liberty to denote the trace of $\varphi_j \in W^{1,2}(\Lambda)$ on $\partial\Lambda$ again by φ_j.

(ii) Partial integration shows that the quadratic form (21) corresponds to imposing Robin boundary conditions $(n \cdot \nabla + \chi)\, \psi|_{\partial\Lambda} = 0$ on functions ψ in the domain of the Laplacian on $\mathrm{L}^2(\Lambda)$. Obviously, Neumann boundary conditions correspond to the special case $\chi = 0$. With the present choice (20) of χ they arise if $U_{\rm per} = 0$ such that $\psi = 1$ or, more generally, if $U_{\rm per}$ is reflection invariant (as was supposed in [**KS86**]).

(iii) Denoting by $\lambda_0(H_\Lambda^\chi(V_\omega)) < \lambda_1(H_\Lambda^\chi(V_\omega)) \leq \lambda_2(H_\Lambda^\chi(V_\omega)) \leq \ldots$ the eigenvalues of $H_\Lambda^\chi(V_\omega)$, the eigenvalue-counting function

$$N\left(E; H_\Lambda^\chi(V_\omega)\right) := \#\left\{n \in \mathbb{N}_0 \,:\, \lambda_n\left(H_\Lambda^\chi(V_\omega)\right) < E\right\} \tag{22}$$

is well-defined for all $\omega \in \Omega$ and all energies $E \in \mathbb{R}$. If $U_{\rm per}$ is bounded from below, it follows from [**Min02**, Thm. 1.3] and (15) that $N(E) = \lim_{|\Lambda|\to\infty}|\Lambda|^{-1}N(E; H_\Lambda^\chi(V_\omega))$. We also refer to [**Min02**] for proofs of some of the above-mentioned properties of the Robin Laplacian.

One important point about the Mezincescu boundary conditions (20) is that the restriction of ψ to Λ continues to be the ground-state eigenfunction of $H_\Lambda^\chi(0)$ with eigenvalue $\lambda_0(H_\Lambda^\chi(0)) = E_0 = 0$. This follows from the fact that ψ satisfies the eigenvalue equation, the boundary conditions and that ψ is strictly positive.

Our proof of Theorem 2.8 is based on the following sandwiching bound on the integrated density of states.

Proposition 3.2. *Let $\Lambda \subset \mathbb{R}^d$ be a d-dimensional open cuboid, which is compatible with the lattice $\mathbb{Z}^d$. Then the integrated density of states N obeys*

$$\begin{aligned} |\Lambda|^{-1}\, \mathbb{P}\Big\{\omega \in \Omega \,:\, \lambda_0\left(H_\Lambda^D(V_\omega)\right) < E\Big\} &\leq N(E) \\ &\leq |\Lambda|^{-1} N\left(E; H_\Lambda^\chi(0)\right)\ \ \mathbb{P}\Big\{\omega \in \Omega \,:\, \lambda_0\left(H_\Lambda^\chi(V_\omega)\right) < E\Big\} \end{aligned} \tag{23}$$

for all energies $E \in \mathbb{R}$.

PROOF. For the lower bound on N, see [**KM83a**, Eq. (4) and (21)] or [**KS86**, Eq. (2)]. The upper bound follows from [**Mez87**, Eq. (29)]. □

Remark 3.3. Since the bracketing [**Mez87**, Prop. 1] [**CL90**, Probl. I.7.19] applies to Robin boundary conditions with more general real $\chi \in \mathrm{L}^\infty(\partial\Lambda)$ than the one defined in (20), the same is true for the upper bound in (23).

3.2. Elementary facts about marginal impurity potentials. Key quantities in our proof of Theorem 2.8 are the marginal impurity potentials $f^{(1)} : \mathbb{R}^{d_1} \to [0,\infty[$ and $f^{(2)} : \mathbb{R}^{d_2} \to [0,\infty[$ for the x_1- and x_2-direction, respectively. For the given $f \in \mathrm{L}^1(\mathbb{R}^d)$ they are defined as follows

$$f^{(1)}(x_1) \quad := \textstyle\int_{\mathbb{R}^{d_2}} f(x_1,x_2)\, dx_2. \tag{24}$$

$$f^{(2)}(x_2) \quad := \textstyle\int_{\mathbb{R}^{d_1}} f(x_1,x_2)\, dx_1 \tag{25}$$

The aim of this Subsection is to collect properties of $f^{(2)}$. Since $f^{(1)}$ results from $f^{(2)}$ by exchanging the role of x_1 and x_2, analogous properties apply to $f^{(1)}$.

Lemma 3.4. *Assumption 2.4 with $m = 2$ implies that there exist two constants $0 < f_1$, $f_2 < \infty$ such that*

$$\frac{f_1}{|x_2|^{\alpha_2(1-\gamma_1)}} \leq \int_{|y_2|<\frac{1}{2}} f^{(2)}(y_2 - x_2)\, dy_2, \qquad f^{(2)}(x_2) \leq \frac{f_2}{|x_2|^{\alpha_2(1-\gamma_1)}} \tag{26}$$

for large enough $|x_2| > 0$.

PROOF. The lemma follows by elementary integration. In doing so, one may replace the maximum norm $|\cdot|$ by the equivalent Euclidean 2-norm in both (12) and (26). □

Lemma 3.5. *Assumption 2.4 with $m = 2$ implies that there exists some constant $0 < f_3 < \infty$ such that*

$$\int_{|x_2|>L} f^{(2)}(x_2)\, dx_2 \leq f_3\, L^{-\alpha_2(1-\gamma)} \tag{27}$$

for sufficiently large $L > 0$.

PROOF. By Lemma 3.4 we have $\int_{|x_2|>L} f^{(2)}(x_2)\, dx_2 \leq f_2 \int_{|x_2|>L} |x_2|^{-\alpha_2(1-\gamma_1)}\, dx_2$ for sufficiently large $L > 0$. The assertion follows by elementary integration and the fact that $\alpha_2(1-\gamma_1) - d_2 = \alpha_2(1-\gamma)$. □

Remark 3.6. One consequence of Lemma 3.5, which will be useful below, is the following inequality

$$\sup_{|y_2|\leq L/2} \int_{|x_2|>L^\beta} f^{(2)}(x_2 - y_2)\, dx_2 \leq f_3\, \left(2/L^\beta\right)^{\alpha_2(1-\gamma)} \tag{28}$$

valid for all $\beta \geq 1$ and sufficiently large $L > 1$. It is obtained by observing that the integral in (28) equals

$$\int_{|x_2+j_2|>L^\beta} f^{(2)}(x_2)\, dx_2 \leq \int_{|x_2|\geq L^\beta/2} f^{(2)}(x_2)\, dx_2. \tag{29}$$

Here the last inequality results from the triangle inequality $|x_2 + y_2| \leq |x_2| + |y_2|$ and the fact that $|y_2| L/2 \leq L^\beta/2$.

4. Upper bound

For an asymptotic evaluation of the upper bound in Proposition 3.2 for small energies E we distinguish the three regimes defined below Theorem 2.8: quantum, quantum-classical and classical.

4.1. Regularisation of random Borel measure. In all of the above mentioned cases it will be necessary to regularise the given random Borel measure μ by introducing a cut off. For this purpose we define a regularised random Borel measure $\mu^{(h)} : \Omega \times \mathcal{B}(\mathbb{R}^d) \to [0, \infty[$ with parameter $h > 0$ by $\mu_\omega^{(h)}(\Lambda) := \sum_{j \in \mathbb{Z}^d} \mu_\omega^{(h)}\big(\Lambda \cap \Lambda_j\big)$ where

$$\mu_\omega^{(h)}\big(\Lambda \cap \Lambda_j\big) := \begin{cases} \mu_\omega\big(\Lambda \cap \Lambda_j\big) & \mu_\omega\big(\Lambda_j\big) \leq h \\ h \, \dfrac{\mu_\omega\big(\Lambda \cap \Lambda_j\big)}{\mu_\omega\big(\Lambda_j\big)} & \text{otherwise} \end{cases} \tag{30}$$

for all $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ and all $\omega \in \Omega$.

Remark 4.1. Since $\mu_\omega^{(h)}(\emptyset) = 0$ and $\mu_\omega^{(h)}\big(\bigcup_n \Lambda^{(n)}\big) = \sum_n \mu_\omega^{(h)}\big(\Lambda^{(n)}\big)$ for any collection of disjoint $\Lambda^{(n)} \in \mathcal{B}(\mathbb{R}^d)$, each realization $\mu_\omega^{(h)}$ is indeed a measure on the Borel sets $\mathcal{B}(\mathbb{R}^d)$. It is locally finite and hence a Borel measure, because $\mu_\omega^{(h)}(\Lambda_j) \leq h$ for all $j \in \mathbb{Z}^d$ and all $\omega \in \Omega$.

For future reference we collect some properties of $\mu^{(h)}$.

Lemma 4.2. *Let $h > 0$. Then the following three assertions hold true:*

(i) $\mu_\omega^{(h)}(\Lambda) \leq \min\big\{\mu_\omega(\Lambda), h \,\#\big\{j \in \mathbb{Z}^d : \Lambda \cap \Lambda_j \neq \emptyset\big\}\big\}$ *for all $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ and all $\omega \in \Omega$.*

(ii) *the intensity measure $\overline{\mu}^{(h)} : \mathcal{B}(\mathbb{R}^d) \to [0, \infty[$ given by $\overline{\mu}^{(h)}(\Lambda) := \mathbb{E}\big[\mu^{(h)}(\Lambda)\big]$ is a Borel measure which is $\mathbb{Z}^d$-periodic and obeys $\overline{\mu}^{(h)}(\Lambda_0) > 0$.*

(iii) *the random variables $\big(\mu^{(h)}(\Lambda_j)\big)_{j \in \mathbb{Z}^d}$ are independent and identically distributed.*

PROOF. The first part of the first assertion is immediate. The other part follows from the monotonicity $\mu_\omega(\Lambda \cap \Lambda_j) \leq \mu_\omega(\Lambda_j) \leq h$ for all $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, $j \in \mathbb{Z}^d$ and all $\omega \in \Omega$. The claimed $\mathbb{Z}^d$-periodicity of the intensity measure is traced back to the $\mathbb{Z}^d$-stationarity of μ. The inequality in the second assertion holds, since $\mu_\omega(\Lambda_0)$ is not identical zero for $\mathbb{P}$-almost all $\omega \in \Omega$ (confer Assumption 2.1). The third assertion follows from the corresponding property of μ (confer Assumption 2.1). $\square$

4.2. Quantum regime. Throughout this subsection we suppose that (qm) holds. Assumption 2.4 on the impurity potential requires the existence of some constant $f_u > 0$ and some Borel set $F \in \mathcal{B}(\mathbb{R}^d)$ with $|F| > 0$ such that

$$f \geq f_u \chi_F. \tag{31}$$

Without loss of generality, we will additionally suppose that $F \subset \Lambda_0$. We start by constructing a lower bound on the lowest Mezincescu eigenvalue $\lambda_0\left(H_\Lambda^\chi(V_\omega)\right)$ showing up in the right-hand side of (23) when choosing the interior of the closure

$$\Lambda := \overline{\bigcup_{|j|<L} \Lambda_j}^{\,\mathrm{int}} \tag{32}$$

of unit cubes, which are at most at a distance $L > 1$ from the origin. By construction, the cube Λ is open and compatible with the lattice.

4.2.1. *Lower bound on the lowest Mezincescu eigenvalue.* From Lemma 4.2(i) and (31) we conclude that the potential $V_{\omega,h} : \mathbb{R}^d \to [0,\infty[$ given by

$$V_{\omega,h}(x) := f_u \int_{\mathbb{R}^d} \chi_F(x-y)\, \mu_\omega^{(h)}(dy) = f_u\, \mu_\omega^{(h)}\big(x - F\big) \tag{33}$$

in terms of the regularised Borel measure $\mu_\omega^{(h)}$, provides a lower bound on V_ω for every $h > 0$ and $\omega \in \Omega$. The fact that the pointwise difference $x - F$ is contained in a cube, which consists of (at most) 3^d unit cubes, together with Lemma 4.2(i) implies the estimate

$$V_{\omega,h}(x) \leq 3^d f_u h \tag{34}$$

for all $\omega \in \Omega$ and all $x \in \mathbb{R}^d$. Taking h small enough thus ensures that the maximum of the potential $V_{\omega,h}$ is smaller than the energy difference of the lowest and the first eigenvalue of $H_\Lambda^\chi(0)$. This enables one to make use of Temple's inequality to obtain a lower bound on the lowest Mezincescu eigenvalue in the quantum regime.

Proposition 4.3. *Let Λ denote the open cube (32). Moreover, let $h := (r_0 L)^{-2}$ with $r_0 > 0$. Then the lowest eigenvalue of $H_\Lambda^\chi(V_{\omega,h})$ is bounded from below according to*

$$\lambda_0\big(H_\Lambda^\chi(V_{\omega,h})\big) \geq \frac{1}{2\,|\Lambda|} \int_\Lambda V_{\omega,h}(x)\, \psi(x)^2\, dx \tag{35}$$

for all $\omega \in \Omega$, all $L > 1$ and large enough $r_0 > 0$. [*Recall the definition of ψ at the beginning of Subsection 3.1.*]

PROOF. By construction $\psi_L := |\Lambda|^{-1/2}\, \psi \in L^2(\Lambda)$ is the normalised ground-state eigenfunction of $H_\Lambda^\chi(0)$ which satisfies $H_\Lambda^\chi(0)\psi_L = 0$. Choosing this function as the variational function in Temple's inequality [**RS78**, Thm. XIII.5] yields the lower bound

$$\lambda_0\big(H_\Lambda^\chi(V_{\omega,h})\big) \geq \big\langle \psi_L, V_{\omega,h}\, \psi_L \big\rangle - \frac{\big\langle V_{\omega,h}\, \psi_L, V_{\omega,h}\, \psi_L \big\rangle}{\lambda_1\big(H_\Lambda^\chi(0)\big) - \big\langle \psi_L, V_{\omega,h}\, \psi_L \big\rangle} \tag{36}$$

provided the denominator in (36) is strictly positive. To check this we note that [**Mez87**, Prop. 4] implies that there is some constant $c_0 > 0$ such that

$$\lambda_1\big(H_\Lambda^\chi(0)\big) = \lambda_1\big(H_\Lambda^\chi(0)\big) - \lambda_0\big(H_\Lambda^\chi(0)\big) \geq 2 c_0 L^{-2} \tag{37}$$

for all $L > 1$. Moreover, we estimate $\big\langle \psi_L, V_{\omega,h}\, \psi_L \big\rangle \leq 3^d f_u\, h \leq c_0 L^{-2}$ for large enough $r_0 > 0$. To bound the numerator in (50) from above, we use the inequality $\big\langle V_{\omega,h}\, \psi_L, V_{\omega,h}\, \psi_L \big\rangle \leq \big\langle \psi_L, V_{\omega,h}\, \psi_L \big\rangle\, 3^d f_u h \leq \big\langle \psi_L, V_{\omega,h}\, \psi_L \big\rangle\, c_0/(2L^2)$ valid for large enough $r_0 > 0$. □

We proceed by constructing a lower bound on the right-hand side of (36). For this purpose we define the cube

$$\widetilde{\Lambda} := \bigcup_{|j| < L-1} \Lambda_j \tag{38}$$

which is contained in the cube Λ defined in (32). In fact it is one layer of unit cubes smaller than Λ.

Lemma 4.4. *There exists a constant $0 < c_1 < \infty$ (which is independent of ω, L and h) such that*

$$\frac{1}{|\Lambda|} \int_\Lambda V_{\omega,h}(x)\, \psi(x)^2\, dx \geq \frac{c_1 h}{|\widetilde{\Lambda}|}\, \#\Big\{ j \in \mathbb{Z}^d \cap \widetilde{\Lambda} \,:\, \mu_\omega\big(\Lambda_j\big) \geq h \Big\} \tag{39}$$

for all $\omega \in \Omega$, all $L > 1$ and all $h > 0$.

PROOF. Pulling out the strictly positive infimum of ψ^2 and using its $\mathbb{Z}^d$-periodicity, we estimate

$$\begin{aligned}\int_\Lambda V_{\omega,h}(x)\,\psi(x)^2\,dx \quad &\geq \quad \inf_{z\in\Lambda_0}\psi(z)^2\,f_u\int_{\mathbb{R}^d}|\Lambda\cap(F+y)|\;\mu_\omega^{(h)}(dy)\\ &\geq \quad \inf_{z\in\Lambda_0}\psi(z)^2\,f_u\,|F|\;\mu_\omega^{(h)}\big(\widetilde{\Lambda}\big)\end{aligned}\tag{40}$$

by omitting positive terms and using Fubini's theorem together with the fact that $F\subset\Lambda_0$. The proof is completed with the help of the inequality

$$\mu_\omega^{(h)}\big(\widetilde{\Lambda}\big)=\sum_{j\in\mathbb{Z}^d\cap\widetilde{\Lambda}}\min\big\{h,\mu_\omega(\Lambda_j)\big\}\geq h\;\#\Big\{j\in\mathbb{Z}^d\cap\widetilde{\Lambda}\,:\,\mu_\omega\big(\Lambda_j\big)\geq h\Big\}\tag{41}$$

and $|\Lambda|\leq 3^d|\widetilde{\Lambda}|$ valid for all $L>1$. $\square$

4.2.2. *Proof of Theorem 2.8 – first part: quantum regime.* We fix $r_0>0$ large enough to ensure the validity of (35) in Proposition 4.3. For a given energy $E>0$ we then pick

$$L:=\left(\frac{c_1}{4r_0^2E}\right)^{1/2}\tag{42}$$

where the constant c_1 has been fixed in Lemma 4.4. Finally, we choose the cube Λ from (32) and set $h:=(r_0L)^{-2}$. Proposition 4.3 and (39) yield the estimate

$$\begin{aligned}&\mathbb{P}\Big\{\omega\in\Omega\,:\,\lambda_0\left(H_\Lambda^\chi(V_\omega)\right)<E\Big\}\\ &\quad\leq\mathbb{P}\left\{\omega\in\Omega\,:\,\#\Big\{j\in\mathbb{Z}^d\cap\widetilde{\Lambda}\,:\,\mu_\omega\big(\Lambda_j\big)\geq h\Big\}<\frac{2E}{c_1h}|\widetilde{\Lambda}|\right\}\\ &\quad=\mathbb{P}\left\{\omega\in\Omega\,:\,\#\Big\{j\in\mathbb{Z}^d\cap\widetilde{\Lambda}\,:\,\mu_\omega\big(\Lambda_j\big)<h\Big\}>\frac{|\widetilde{\Lambda}|}{2}\right\}.\end{aligned}\tag{43}$$

Here the last equality uses the fact that $h=4E/c_1$. In case $\overline{\mu}(\Lambda_j)>h$, that is, for sufficiently small E, the right-hand side is the probability of a large deviation event [**DZ98**]. Consequently (confer [**KS86**, Prop. 4]), there exists a constant $0<c_2<\infty$, such that (43) is estimated from above by

$$\exp\Big[-c_2|\widetilde{\Lambda}|\Big]\leq\exp\big[-c_2n_uL^d\big]=\exp\Big[-c_3E^{-d/2}\Big]\tag{44}$$

Here the inequality follows from the estimate $|\widetilde{\Lambda}|\geq n_uL^d$ for some constant $n_u>0$ and all $L>2$. The existence of a constant $c_3>0$ ensuring the validity of the last equality follows from (42). Inserting this estimate in the right-hand side of (23) completes the first part of the proof of Theorem 2.8 for the quantum-classical regime, since the pre-factor in the upper bound in Proposition 3.2 is negligible. $\square$

4.3. Quantum-classical regime. Without loss of generality we suppose that (qm/cl) holds throughout this subsection, that is $d_1/2\geq\gamma_1/(1-\gamma)$ and $d_2/2<\gamma_2/(1-\gamma)$. We start by constructing a lower bound on the lowest Mezincescu eigenvalue $\lambda_0\left(H_\Lambda^\chi(V_\omega)\right)$ showing up in the right-hand side of (23) when choosing

$$\Lambda:=\overline{\bigcup_{|j_1|<L}\Lambda_{(j_1,0)}}^{\,\mathrm{int}}\tag{45}$$

a cuboid with some $L>1$. By construction it is open and compatible with the lattice.

4.3.1. *Lower bound on the lowest Mezincescu eigenvalue.* From Lemma 4.2(i) we conclude that for every $R > 0$ and $\omega \in \Omega$ the potential $V_{\omega,R} : \mathbb{R}^d \to [0,\infty[$ given by

$$V_{\omega,R}(x) := \int_{|y_2|>R} f(x-y)\, \mu_\omega^{(1)}(dy) \tag{46}$$

in terms of the regularised Borel measure $\mu_\omega^{(1)}$, provides a lower bound on V_ω. Therefore $\lambda_0\left(H_\Lambda^\chi(V_\omega)\right) \geq \lambda_0\big(H_\Lambda^\chi(V_{\omega,R})\big)$. It will be useful to collect some facts related to $V_{\omega,R}$.

Lemma 4.5. *Let $R > 1$ and define $V_R : \mathbb{R}^d \to [0,\infty[$ by*

$$V_R(x) := \sum_{\substack{j_1 \in \mathbb{Z}^{d_1} \\ |j_2| > R-1}} \sup_{y \in \Lambda_j} f(x-y). \tag{47}$$

Then the following three assertions hold true:

(i) $V_{\omega,R} \leq V_R$ *for every* $\omega \in \Omega$.

(ii) V_R *is* $\mathbb{Z}^{d_1}$*-periodic with respect to translations in the* x_1*-direction.*

(iii) there exists some constant $c > 0$ *such that* $\sup_{x\in\Lambda_0} V_R(x) \leq c\, R^{-\alpha_2(1-\gamma)}$ *for large enough* $R > 1$.

PROOF. The first assertion follows from the inequalities

$$V_{\omega,R}(x) \leq \sum_{\substack{j_1 \in \mathbb{Z}^{d_1} \\ |j_2| > R-1}} \int_{\Lambda_j} f(x-y)\, \mu_\omega^{(1)}(dy) \tag{48}$$

and $\mu_\omega^{(1)}(\Lambda_j) \leq 1$ valid for all $\omega \in \Omega$. The second assertion holds true by definition. The third assertion derives from (12) and is the “summation” analogue of Lemma 3.5. □

The cut-off R guarantees that the potential $V_{\omega,R}$ does not exceed a certain value. In particular, taking R large enough ensures that this value is smaller than the energy difference of the lowest and the first eigenvalue of $H_\Lambda^\chi(0)$. This enables one to make use of Temple’s inequality to obtain a lower bound on the lowest Mezincescu eigenvalue in the quantum-classical regime.

Proposition 4.6. *Let Λ denote the cuboid (45). Moreover, let $R := (r_0 L)^{2/\alpha_2(1-\gamma)}$ with $r_0 > 0$. Then the lowest eigenvalue of $H_\Lambda^\chi(V_{\omega,R})$ is bounded from below according to*

$$\lambda_0\big(H_\Lambda^\chi(V_{\omega,R})\big) \geq \frac{1}{2\,|\Lambda|} \int_\Lambda V_{\omega,R}(x)\, \psi(x)^2\, dx \tag{49}$$

for all $\omega \in \Omega$, all $L > 1$ and large enough $r_0 > 0$. [*Recall the definition of ψ at the beginning of Subsection 3.1.*]

PROOF. The proof parallels the one of Proposition 4.6. By construction $\psi_L := |\Lambda|^{-1/2}\, \psi \in L^2(\Lambda)$ is the normalised ground-state eigenfunction of $H_\Lambda^\chi(0)$ which satisfies $H_\Lambda^\chi(0)\psi_L = 0$. Choosing this function as the variational function in Temple’s inequality [**RS78**, Thm. XIII.5] yields the lower bound

$$\lambda_0\big(H_\Lambda^\chi(V_{\omega,R})\big) \geq \big\langle \psi_L, V_{\omega,R}\, \psi_L \big\rangle - \frac{\big\langle V_{\omega,R}\, \psi_L, V_{\omega,R}\, \psi_L \big\rangle}{\lambda_1\big(H_\Lambda^\chi(0)\big) - \big\langle \psi_L, V_{\omega,R}\, \psi_L \big\rangle} \tag{50}$$

provided the denominator in (50) is strictly positive. To check this we note that a simple extension of [**Mez87**, Prop. 4] from cubes to cuboids implies that there is some constant

$c_0 > 0$ such that $\lambda_1\big(H_\Lambda^\chi(0)\big) = \lambda_1\big(H_\Lambda^\chi(0)\big) - \lambda_0\big(H_\Lambda^\chi(0)\big) \geq 2c_0 L^{-2}$ for all $L > 1$. Moreover, using Lemma 4.5 and the definition of R we estimate

$$\big\langle \psi_L, V_{\omega,R}\,\psi_L\big\rangle \leq \big\langle \psi_L, V_R\,\psi_L\big\rangle = \int_{\Lambda_0} V_R(x)\,\psi(x)^2 dx \leq c\,\big(r_0 L\big)^{-2} \leq c_0 L^{-2} \tag{51}$$

for large enough $r_0 > 0$. To bound the numerator in (50) from above, we use the inequality $\big\langle V_{\omega,R}\,\psi_L, V_{\omega,R}\,\psi_L\big\rangle \leq \big\langle \psi_L, V_{\omega,R}\,\psi_L\big\rangle \sup_{x\in\Lambda} V_R(x)$. Lemma 4.5 ensures that $\sup_{x\in\Lambda} V_R(x) = \sup_{x\in\Lambda_0} V_R(x)$ and thus yields the bound

$$\big\langle V_{\omega,R}\,\psi_L, V_{\omega,R}\,\psi_L\big\rangle \leq \big\langle \psi_L, V_{\omega,R}\,\psi_L\big\rangle\, c\,\big(r_0 L\big)^{-2} \leq \big\langle \psi_L, V_{\omega,R}\,\psi_L\big\rangle \frac{c_0}{2} L^{-2} \tag{52}$$

for large enough $r_0 > 0$. □

We proceed by constructing a lower bound on the right-hand side of (49). For this purpose we set

$$\widetilde{\Lambda} := \bigcup_{\substack{|j_1|\leq L/8 \\ R<|j_2|\leq 2R}} \Lambda_j \tag{53}$$

a union of disjoint cuboids.

Lemma 4.7. *There exist two constants* $0 < c_2, c_3 < \infty$ *(which are independent of* ω*,* L *and* R*) such that*

$$\int_\Lambda V_{\omega,R}(x)\,\psi(x)^2\,dx \geq \frac{c_2}{R^{\alpha_2(1-\gamma_1)}}\;\mu_\omega^{(1)}\big(\widetilde{\Lambda}\big) - c_3\,|\Lambda|\,L^{-\alpha_1(1-\gamma)} \tag{54}$$

for all $\omega \in \Omega$ *and large enough* $L > 1$ *and* $R > 1$.

Remark 4.8. An important consequence of this lemma reads as follows. There exists some constant $n_u > 0$ such that the number of lattice points in $\widetilde{\Lambda}$ is estimated from below by $|\widetilde{\Lambda}| \geq n_u\,|\Lambda| R^{d_2}$ for all $L > 1$ and $R > 1$ and some constant $n_u > 0$. Therefore $|\widetilde{\Lambda}|/(|\Lambda| R^{\alpha_2(1-\gamma_1)}) \geq n_u / R^{\alpha_2(1-\gamma)}$. Choosing $R = (r_0 L)^{2/\alpha_2(1-\gamma)}$ as in Proposition 4.6, we thus arrive at the lower bound

$$\frac{1}{|\Lambda|}\int_\Lambda V_{\omega,R}(x)\,\psi(x)^2\,dx \geq \frac{c_2\,n_u}{(r_0 L)^2}\;\frac{1}{|\widetilde{\Lambda}|}\sum_{j\in\mathbb{Z}^d\cap\widetilde{\Lambda}} \mu_\omega^{(1)}\big(\Lambda_j\big) - c_3\,L^{-\alpha_1(1-\gamma)} \tag{55}$$

valid for all $r_0 > 0$ and large enough $L > 1$.

PROOF OF LEMMA 4.7. Pulling out the strictly positive infimum of ψ^2 and using its $\mathbb{Z}^d$-periodicity, we estimate

$$\begin{aligned}\int_\Lambda V_{\omega,R}(x)\,\psi(x)^2\,dx &\geq \inf_{z\in\Lambda_0} \psi(z)^2 \int_\Lambda V_{\omega,R}(x)\,dx \\ &\geq \inf_{z\in\Lambda_0} \psi(z)^2 \int_{\widetilde{\Lambda}} \left(\int_\Lambda f(x-y)\,dx\right) \mu_\omega^{(1)}(dy)\end{aligned} \tag{56}$$

by omitting positive terms and using Fubini's theorem. The inner integral in the last line is estimated from below with the help of Lemma 3.4 in terms of the marginal impurity potential $f^{(2)}$ (recall definition (25)) according to

$$\begin{aligned}\int_\Lambda f(x-y)\,dx &= \int_{|x_2|<\frac{1}{2}} f^{(2)}(x_2-y_2)\,dx_2 - \sum_{|k_1|\geq L}\int_{\Lambda_{(k_1,0)}} f(x-y)\,dx \\ &\geq \frac{f_1}{(2R+1)^{\alpha_2(1-\gamma_1)}} - \sum_{|k_1|\geq L}\int_{\Lambda_0} f\big(x+(k_1,0)-y\big)\,dx\end{aligned} \tag{57}$$

for all $|y_2| \leq 2R+1$ and large enough $R > 0$. The first term on the right-hand side yields the first term on the right-hand side of (54). To estimate the remainder we decompose the y-integration of the second term in (57) with respect to $\mu_\omega^{(1)}$ and use the fact that $\mu_\omega^{(1)}(\Lambda_j) \leq 1$. This yields an estimate of the form

$$\begin{aligned} \int_{\widetilde{\Lambda}} \left(\int_{\Lambda_0} g(x-y)\, dx \right) \mu_\omega^{(1)}(dy) \quad &\leq \quad \sum_{j \in \mathbb{Z}^d \cap \widetilde{\Lambda}} \sup_{y \in \Lambda_0} \int_{\Lambda_0} g(x-y-j)\, dx \\ &\leq \quad 3^d \sum_{\substack{|j_1| \leq L/2 \\ j_2 \in \mathbb{Z}^{d_2}}} \int_{\Lambda_0} g(x-j)\, dx \\ &= \quad 3^d \sum_{|j_1| \leq L/4} \int_{|x_1|<1/2} g^{(1)}(x_1 - j_1)\, dx_1 \end{aligned} \tag{58}$$

valid for all $g \in \mathrm{L}^1(\mathbb{R}^d)$. Here the second inequality holds for every $L \geq 8$ (so that $L/4 - L/8 \geq 1$) and follows from enlargening the j_2-summation and the fact that the pointwise difference $\Lambda_0 - \Lambda_0$ is contained in the cube centred at the origin and consisting of 3^d unit cubes. The last equality uses the definition (24) for a marginal impurity potential. Substituting $g(x) = f(x + (k_1, 0))$ in the above chain of inequalities, performing the k_1-summation and enlargening the x_1-integration thus yields

$$3^d \sum_{|j_1| \leq L/4} \int_{|x_1|>L/2} f^{(1)}(x_1 - j_1)\, dx_1 \leq 3^d n_0 |\Lambda| \sup_{|j_1| \leq L/4} \int_{|x_1|>L/2} f^{(1)}(x_1 - j_1)\, dx_1 \tag{59}$$

as an upper bound for the remainder for all $L \geq 8$. Here the inequality follows from the estimate $\#\{|j_1| \leq L/2\} \leq n_0|\Lambda|$ for some $n_0 < \infty$ and all $L > 1$. The proof is completed by employing a result for $f^{(1)}$ analogous to (28). $\square$

4.3.2. *Proof of Theorem 2.8 – first part: quantum-classical regime.* We fix $r_0 > 1/(2\,\overline{\mu}^{(1)}(\Lambda_0))$ large enough to ensure the validity of (49) in Proposition 4.6. For a given energy $E > 0$ we then pick

$$L := \left(\frac{c_2\, n_u}{2 r_0^3 E} \right)^{1/2} \tag{60}$$

where the constants c_2 and n_u have been fixed in Lemma 4.7 and Remark 4.8. Finally, we choose the cuboid Λ from (45) and set $R := (r_0 L)^{2/\alpha_2(1-\gamma)}$. Proposition 4.6 and (55) then yield the estimate

$$\begin{aligned} &\mathbb{P}\Big\{ \omega \in \Omega \,:\, \lambda_0\left(H_\Lambda^\chi(V_\omega)\right) < E \Big\} \\ &\leq \mathbb{P}\left\{ \omega \in \Omega \,:\, \frac{1}{|\widetilde{\Lambda}|} \sum_{j \in \mathbb{Z}^d \cap \widetilde{\Lambda}} \mu_\omega^{(1)}\big(\Lambda_j\big) < \frac{(r_0 L)^2}{c_2 n_u} \left(2E + c_2 L^{-\alpha_1(1-\gamma)} \right) \right\} \\ &\leq \mathbb{P}\left\{ \omega \in \Omega \,:\, \frac{1}{|\widetilde{\Lambda}|} \sum_{j \in \mathbb{Z}^d \cap \widetilde{\Lambda}} \mu_\omega^{(1)}\big(\Lambda_j\big) < \frac{2}{r_0} \right\} \end{aligned} \tag{61}$$

provided $E > 0$ is small enough, equivalently L is large enough. Here the last inequality results from (60) and from the first inequality in (qm/cl), which implies that $c_3 r_0^3 L^2 \leq c_2 n_u L^{\alpha_1(1-\gamma)}$ for large enough $L > 0$. Since $2/r_0 \leq \overline{\mu}^{(1)}(\Lambda_0)$ by assumption on r_0, the right-hand side of (61) is the probability of a large-deviation event [**Dur96, DZ98**].

Consequently, there exists some constant $c_4 > 0$ (which is independent of L) such that (61) is estimated from above by

$$\begin{aligned}\exp\Big[-c_4\,|\widetilde{\Lambda}|\Big] &\leq \exp\Big[-c_4\,n_u\,L^{d_1}\,(r_0 L)^{2\gamma_2/(1-\gamma)}\Big]\\ &= \exp\Big[-c_5\,E^{-d_1/2-\gamma_2/(1-\gamma)}\Big]. \qquad (62)\end{aligned}$$

Here the existence of a constant $c_5 > 0$ ensuring the validity of the last equality follows from (60). Inserting this estimate in the right-hand side of (23) completes the first part of the proof of Theorem 2.8 for the quantum-classical regime, since the pre-factor in the upper bound in Proposition 3.2 is negligible. □

4.4. Classical regime. Throughout this Subsection we suppose that (cl) holds. For an asymptotic evaluation of the upper bound in Proposition 3.2 in the present case, we define

$$\beta_k := \frac{2}{d_k}\,\frac{\gamma_k}{1-\gamma} = \frac{2}{\alpha_k\,(1-\gamma)}, \qquad k \in \{1,2\} \tag{63}$$

and construct a lower bound on the lowest Mezincescu eigenvalue $\lambda_0\big(H^{\chi}_{\Lambda_0^{\rm int}}(V_\omega)\big)$ showing up in the right-hand side of (23) when choosing $\Lambda = \Lambda_0^{\rm int}$ the open unit cube there.

4.4.1. *Lower bound on the lowest Mezincescu eigenvalue.* For every $L > 1$ and $\omega \in \Omega$ the potential $V_{\omega,L} : \mathbb{R}^d \to [0,\infty[$ given by

$$V_{\omega,L}(x) := \int_{\substack{|y_1|>L^{\beta_1}\\ |y_2|>L^{\beta_2}}} f(x-y)\,\mu_\omega^{(1)}(dy) \tag{64}$$

in terms of the regularised Borel measure $\mu_\omega^{(1)}$, provides a lower bound on V_ω. Therefore $\lambda_0\big(H^{\chi}_{\Lambda_0^{\rm int}}(V_\omega)\big) \geq \lambda_0\big(H^{\chi}_{\Lambda_0^{\rm int}}(V_{\omega,L})\big)$. It will be useful to collect some facts related to $V_{\omega,L}$.

Lemma 4.9. *Let $L > 1$ and define $V_L : \mathbb{R}^d \to [0,\infty[$ by*

$$V_L(x) := \sum_{\substack{|j_1|>L^{\beta_1}-1\\ |j_2|>L^{\beta_2}-1}} \sup_{y\in\Lambda_j} f(x-y). \tag{65}$$

Then we have $V_{\omega,L} \leq V_L$ for every $\omega \in \Omega$. Moreover, the supremum $\sup_{x\in\Lambda_0} V_L(x)$ is arbitrarily small for large enough $L > 1$.

PROOF. The first assertion follows analogously as in Lemma 4.5. The second one derives from the second inequality in (12). □

Remark 4.10. It is actually not difficult to prove that there exists some constant $0 < C < \infty$ (which is independent of L) such that $\sup_{x\in\Lambda_0} V_L(x) \leq C\,L^{-2}$ for large enough $L > 0$.

The next proposition contains the key estimate on the lowest Mezincescu eigenvalue in the classical regime. In contrast to the quantum-classical regime, the specific choice of the cut-off made in (64) is irrelevant as far as the applicability of Temple's inequality in the subsequent Proposition is concerned. The chosen length scales L^{β_1} and L^{β_2} will rather become important later on.

Proposition 4.11. *Let $\Lambda_0^{\rm int}$ be the open unit cube. Then the lowest eigenvalue of $H^{\chi}_{\Lambda_0^{\rm int}}(V_{\omega,L})$ is bounded from below according to*

$$\lambda_0\big(H^{\chi}_{\Lambda_0^{\rm int}}(V_{\omega,L})\big) \geq \frac{1}{2}\int_{\Lambda_0} V_{\omega,L}(x)\,\psi(x)^2\,dx \tag{66}$$

for all $\omega \in \Omega$ and large enough $L > 1$. [*Recall the definition of ψ at the beginning of Subsection 3.1.*]

PROOF. The proof again parallels that of Proposition 4.3. In a slight abuse of notation, let ψ denote the restriction of ψ to Λ_0^{int} throughout this proof. Temple's inequality [**RS78**, Thm. XIII.5] together with the fact that $H^{\chi}_{\Lambda_0^{\mathrm{int}}}(0)\psi = 0$ yields the lower bound

$$\lambda_0\big(H^{\chi}_{\Lambda_0^{\mathrm{int}}}(V_{\omega,L})\big) \geq \langle \psi, V_{\omega,L}\,\psi\rangle - \frac{\langle V_{\omega,L}\,\psi, V_{\omega,L}\,\psi\rangle}{\lambda_1\big(H^{\chi}_{\Lambda_0^{\mathrm{int}}}(0)\big) - \langle \psi, V_{\omega,L}\,\psi\rangle} \tag{67}$$

provided that the denominator is strictly positive. To check this we employ Lemma 4.9 and take $L > 1$ large enough such that $\langle \psi, V_{\omega,L}\,\psi\rangle \leq \lambda_1\big(H^{\chi}_{\Lambda_0^{\mathrm{int}}}(0)\big)/2$. (Note that $\lambda_1\big(H^{\chi}_{\Lambda_0^{\mathrm{int}}}(0)\big)$ is independent of L.) To estimate the numerator in (67) from above, we use the bound $\langle V_{\omega,L}\,\psi, V_{\omega,L}\,\psi\rangle \leq \langle \psi, V_{\omega,L}\,\psi\rangle\, \sup_{x\in\Lambda_0} V_L(x)$. Together with Lemma 4.9 this yields $\langle V_{\omega,L}\,\psi, V_{\omega,L}\,\psi\rangle \leq \langle \psi, V_{\omega,L}\,\psi\rangle \lambda_1\big(H^{\chi}_{\Lambda_0^{\mathrm{int}}}(0)\big)/4$ for large enough $L > 1$. $\square$

Remark 4.12. The simple lower bound $\lambda_0\big(H^{\chi}_{\Lambda_0^{\mathrm{int}}}(V_{\omega,L})\big) \geq \inf_{x\in\Lambda_0} V_{\omega,L}(x)$, which was employed in [**KS86**], would yield a result similar to (72) below, but at the price of assuming that the lower bound in (12) holds pointwise.

We proceed by constructing a lower bound on the right-hand side of (66). For this purpose we set

$$\widetilde{\Lambda} := \bigcup_{\substack{2L^{\beta_1} < |j_1| \leq 4L^{\beta_1} \\ 2L^{\beta_2} < |j_2| \leq 4L^{\beta_2}}} \Lambda_j \tag{68}$$

an annulus-shaped region.

Lemma 4.13. *There exists a constant $c_6 > 0$ (which is independent of ω and L) such that*

$$\int_{\Lambda_0} V_{\omega,L}(x)\,\psi(x)^2\,dx \geq \frac{c_6}{L^{2/(1-\gamma)}}\;\mu_\omega^{(1)}\big(\widetilde{\Lambda}\big) \tag{69}$$

for large enough $L > 0$.

PROOF. Pulling out the strictly positive infimum of ψ^2, using Fubini's theorem and omitting a positive term, we estimate

$$\int_{\Lambda_0} V_{\omega,L}(x)\,\psi(x)^2\,dx \geq \inf_{z\in\Lambda_0} \psi(z)^2 \int_{\widetilde{\Lambda}} \left(\int_{\Lambda_0} f(x-y)\,dx\right) \mu_\omega^{(1)}(dy). \tag{70}$$

Assumption 2.4 implies that the estimate $\int_{\Lambda_0} f(x-y)\,dx \geq f_u/\big[(3L^{\beta_1})^{\alpha_1} + (3L^{\beta_2})^{\alpha_2}\big]$ holds for all $y \in \widetilde{\Lambda}$ and large enough $L > 1$. This completes the proof, since $\alpha_k\beta_k = 2/(1-\gamma)$ for both $k \in \{1,2\}$. $\square$

Remark 4.14. There exists some constant $n_u > 0$ such that the number of lattice points in $\widetilde{\Lambda}$ can be bounded from below according to $|\widetilde{\Lambda}| \geq n_u L^{\beta_1 d_1 + \beta_2 d_2} = n_u L^{2\gamma/(1-\gamma)}$ for all $L > 1$. Lemma 4.13 thus implies the inequality

$$\int_{\Lambda_0} V_{\omega,L}(x)\,\psi(x)^2\,dx \geq \frac{c_6\,n_u}{L^2}\;|\widetilde{\Lambda}|^{-1}\,\mu_\omega^{(1)}\big(\widetilde{\Lambda}\big) \tag{71}$$

for large enough $L > 1$.

4.4.2. *Proof of Theorem 2.8 – first part: classical regime.* For a given energy $E > 0$ we let $L := \big(c_6 n_u \,\overline{\mu}^{(1)}(\Lambda_0)/4E\big)^{1/2}$, where the constant c_6 and n_u have been fixed in Lemma 4.13 and Remark 4.14. Proposition 4.11 and Equation (71) then yield the estimate

$$\mathbb{P}\Big\{\omega \in \Omega \,:\, \lambda_0\left(H^{\chi}_{\Lambda_0^{\rm int}}(V_\omega)\right) < E\Big\} \leq \mathbb{P}\left\{\omega \in \Omega \,:\, \frac{1}{|\widetilde{\Lambda}|} \sum_{j \in \mathbb{Z}^d \cap \widetilde{\Lambda}} \mu_\omega^{(1)}\big(\Lambda_j\big) < \frac{2E\,L^2}{c_6\,n_u}\right\} \tag{72}$$

provided $E > 0$ is small enough, equivalently L is large enough. Since $2EL^2/c_6 n_u = \mathbb{E}\big[\mu_\omega^{(1)}(\Lambda_0)\big]/2$ and the random variables are independent and identically distributed, the last probability is that of a large deviation event [**Dur96, DZ98**]. Consequently, there exists some $c_7 > 0$ such that the right-hand side of (72) is bounded from above by

$$\begin{aligned} \exp\Big[-c_7\,|\widetilde{\Lambda}|\Big] \quad &\leq \quad \exp\Big[-c_7 n_u L^{2\gamma/(1-\gamma)}\Big] \\ &= \quad \exp\Big[-c_7 n_u \left(c_6 n_u \,\overline{\mu}^{(1)}(\Lambda_0)/4E\right)^{\gamma/(1-\gamma)}\Big]. \end{aligned} \tag{73}$$

Since the pre-factor in the upper bound in Proposition 3.2 is negligible, inserting (72) together with (73) in the right-hand side of (23) completes the first part of the proof of Theorem 2.8 for the classical regime. □

5. Lower bound

To complete the proof of Theorem 2.8, it remains to asymptotically evaluate the lower bound in Proposition 3.2 for small energies. This is the topic of the present Section. In order to do so, we first construct an upper bound on the lowest Dirichlet eigenvalue showing up in the left-hand side of (23) when choosing

$$\Lambda := \overline{\bigcup_{|j|<L/4} \Lambda_j}^{\,\rm int} \tag{74}$$

with $L > 0$ there. By construction Λ is open and compatible with the lattice.

5.1. Upper bound on lowest Dirichlet eigenvalue. The following lemma basically repeats [**KS86**, Prop. 5] and its corollary.

Lemma 5.1. *Let Λ denote the open cube (74). There exist two constant $0 < C_1, C_2 < \infty$ (which are independent of ω and L) such that*

$$\lambda_0\left(H^D_\Lambda(V_\omega)\right) \leq C_1\,|\Lambda|^{-1} \int_\Lambda V_\omega(x)\,dx + C_2\,L^{-2} \tag{75}$$

for all $\omega \in \Omega$ and all $L > 1$.

PROOF. We let $\theta \in \mathcal{C}_c^\infty(\Lambda_0)$ denote a smoothed indicator function of the cube $\{x \in \mathbb{R}^d : |x| < 1/4\} \subset \Lambda_0$ and set $\theta_L(x) := \theta\big(x/|\Lambda|^{1/d}\big)$ for all $x \in \Lambda$. Choosing the product of $\theta_L \in \mathcal{C}_c^\infty(\Lambda)$ and the ground-state function ψ of $H(0)$ as the variational function in the Rayleigh-Ritz principle we obtain

$$\begin{aligned} \lambda_0\left(H^D_\Lambda(V_\omega)\right)\,\big\langle\theta_L\psi, \theta_L\psi\big\rangle &\leq \big\langle\theta_L\psi, H^D_\Lambda(V_\omega)\theta_L\psi\big\rangle \\ &= \big\langle\theta_L\psi, V_\omega\theta_L\psi\big\rangle + \big\langle\,(\nabla\theta_L)\,\psi, (\nabla\theta_L)\,\psi\big\rangle \\ &\leq \sup_{y\in\Lambda_0} \psi(y)^2\,\left[\int_\Lambda V_\omega(x)\,dx + |\Lambda|^{1-2/d}\int_{\Lambda_0} |\nabla\theta(x)|^2 dx\right]. \end{aligned} \tag{76}$$

Here the equality uses $H_\Lambda^\chi(0)\psi = 0$ and integration by parts. Observing that $\langle \theta_L\psi, \theta_L\psi\rangle \geq 2^{-d}|\Lambda| \inf_{x\in\Lambda_0} \psi(x)^2$ and that the is some constant $C > 0$ such that $|\Lambda|^{1/d} \geq CL$ for all $L > 1$, completes the proof. $\square$

Our next task is to bound the integral in the right-hand side of (75) from above. For this purpose it will be useful to introduce the cuboid

$$\widetilde{\Lambda} := \bigcup_{\substack{|j_1|\leq 2L^{\beta_1}\\ |j_2|\leq 2L^{\beta_2}}} \Lambda_j, \tag{77}$$

which contains the cube Λ defined in (74). Here and in the following we use the abbreviation $\beta_k := \max\{1, 2/\alpha_k(1-\gamma)\} = 2/d_k \, \max\{d_k/2, \gamma_k/(1-\gamma)\}$, for $k \in \{1,2\}$.

Lemma 5.2. *Let $L > 0$ and define the random variable*

$$W_\omega(L) := |\Lambda|^{-1} \int_{\mathbb{R}^d\setminus\widetilde{\Lambda}} \left(\int_\Lambda f(x-y)\, dx \right) \mu_\omega(dy). \tag{78}$$

Then the following three assertions hold true:

(i) $|\Lambda|^{-1} \int_\Lambda V_\omega(x)\, dx \leq \|f\|_1 \, \mu_\omega\big(\widetilde{\Lambda}\big) + W_\omega(L)$ *for all $\omega \in \Omega$ and all $L > 0$.*

(ii) there exists some constant $0 < C_3 < \infty$ (which is independent of ω and L) such that

$$\mathbb{P}\left\{\omega \in \Omega \,:\, W_\omega(L) \geq C_3 L^{-2}\right\} \leq \frac{1}{2} \tag{79}$$

for large enough L.

(iii) the random variables $\mu\big(\widetilde{\Lambda}\big)$ and $W(L)$ are independent for all $L > 0$.

PROOF. For a proof of the first assertion we decompose the domain of integration and use Fubini's theorem to obtain

$$\begin{aligned}\int_\Lambda V_\omega(x)\, dx &= \int_{\widetilde{\Lambda}} \left(\int_\Lambda f(x-y)\, dx \right) \mu_\omega(dy) + \int_{\mathbb{R}^d\setminus\widetilde{\Lambda}} \left(\int_\Lambda f(x-y)\, dx \right) \mu_\omega(dy) \\ &\leq \|f\|_1 \, \mu_\omega\big(\widetilde{\Lambda}\big) + |\Lambda|\, W_\omega(L).\end{aligned} \tag{80}$$

Here the inequality results from the estimate $\int_\Lambda f(x-y)\, dx \leq \int_{\mathbb{R}^d} f(x)\, dx =: \|f\|_1$ valid for all $y \in \mathbb{R}^d$. This yields Lemma 5.2(i) since $1 \leq |\Lambda|$. For a proof of the second assertion, we employ Chebychev's inequality

$$\begin{aligned}\mathbb{P}\Big\{\omega \in \Omega \,:\, W_\omega(L) \geq C_3 L^{-2}\Big\} &\leq \frac{L^2}{C_3} |\Lambda|^{-1}\, \mathbb{E}\left[\int_{\mathbb{R}^d\setminus\widetilde{\Lambda}} \left(\int_\Lambda f(x-y)\, dx \right) \mu(dy) \right] \\ &= \frac{L^2}{C_3} |\Lambda|^{-1} \int_\Lambda \left(\int_{\mathbb{R}^d\setminus\widetilde{\Lambda}} f(x-y)\, dx \right) \overline{\mu}(dy) \\ &\leq \frac{L^2}{C_3} \overline{\mu}(\Lambda_0) \sup_{y\in\Lambda} \int_{\mathbb{R}^d\setminus\widetilde{\Lambda}} f(x-y)\, dx.\end{aligned} \tag{81}$$

Here the inequality uses the fact that the intensity measure $\overline{\mu}$ is $\mathbb{Z}^d$-periodic. The inner integral is in turn estimated from above in term of two integrals involving the marginal

impurity potentials $f^{(1)}$ and $f^{(2)}$ (recall the definitions (24) and (25))

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\widetilde{\Lambda}} f(x-y)\,dx &\leq \int_{|x_1|>L^{\beta_1}} f^{(1)}(x_1-y_1)\,dx_1 + \int_{|x_2|>L^{\beta_2}} f^{(2)}(x_2-y_2)\,dx_2 \\ &\leq CL^{-2}. \end{aligned} \tag{82}$$

Here the existence of some $0<C<\infty$ ensuring the last inequality for all $|y|\leq L/2$ (that is in particular; for all $y\in\Lambda$) and sufficiently large $L\geq 4$ follows from (28) and the fact that $\beta_k\alpha_k(1-\gamma)\leq 2$. Taking C_3 in (81) large enough yields the second assertion. The third assertion is a consequence of Assumption 2.1(ii). $\square$

5.2. Proof of Theorem 2.8 – final parts. For a given energy $E>0$ we choose

$$L := \left(\frac{3\max\{C_2,C_3\}}{E}\right)^{1/2}, \tag{83}$$

where the constants C_2 and C_3 were fixed in Lemma 5.1 and Lemma 5.2, respectively. Moreover, we pick the cube Λ from (74) and the cuboid $\widetilde{\Lambda}$ from (77). Employing Lemma 5.1 and Lemma 5.2 we estimate the probability in the right-hand side of (23) according to

$$\begin{aligned}&\mathbb{P}\left\{\omega\in\Omega\,:\,\lambda_0\left(H^D_\Lambda(V_\omega)\right)<E\right\}\\ &\geq \mathbb{P}\Big(\left\{\omega\in\Omega\,:\,\lambda_0\left(H^D_\Lambda(V_\omega)\right)<E\right\}\cap\left\{\omega\in\Omega\,:\,W_\omega(L)<C_3L^{-2}\right\}\Big)\\ &\geq \mathbb{P}\left(\left\{\omega\in\Omega\,:\,\mu_\omega\big(\widetilde{\Lambda}\big)<\frac{\max\{C_2,C_3\}\,L^{-2}}{C_1\,\|f\|_1}\right\}\cap\left\{\omega\in\Omega\,:\,W_\omega(L)<C_3L^{-2}\right\}\right).\end{aligned} \tag{84}$$

Since the random variables $\mu(\widetilde{\Lambda})$ and $W(L)$ are independent, the probability in (84) factorises. Thanks to (79) the probability of the second event is bounded from below by $1/2$ provided that L is large enough, equivalently, that $E>0$ is small enough. Employing the decomposition (74) of $\widetilde{\Lambda}$ into $|\widetilde{\Lambda}|$ unit cubes of the lattice $\mathbb{Z}^d$, we have $\mu_\omega\big(\widetilde{\Lambda}\big) = \sum_{j\in\widetilde{\Lambda}\cap\mathbb{Z}^d}\mu_\omega\big(\Lambda_j\big)$ such that the probability of the first event in (84) is bounded from below by

$$\mathbb{P}\left\{\omega\in\Omega\,:\,\mu_\omega\big(\Lambda_j\big)<\frac{\max\{C_2,C_3\}\,L^{-2}}{C_1\,\|f\|_1\,|\widetilde{\Lambda}|}\quad\text{for all } j\in\widetilde{\Lambda}\cap\mathbb{Z}^d\right\}. \tag{85}$$

By construction of $\widetilde{\Lambda}$ there is some constant $n_0>0$ such that $|\widetilde{\Lambda}|\leq n_0\,L^{\beta_1d_1+\beta_2d_2}$. Abbreviating $C_4:=\max\{C_2,C_3\}/(C_1\|f\|_1\,n_0)$ and $\vartheta:=2+\beta_1d_1+\beta_2d_2$, and using the fact that the random variables $\mu(\Lambda_j)$ are independent and identically distributed (by virtue of Assumption 2.1), the last expression (85) may be bounded from below by

$$\begin{aligned}\mathbb{P}\Big\{\omega\in\Omega\,:\,\mu_\omega(\Lambda_0)<C_4L^{-\vartheta}\Big\}^{n_0\,L^{\beta_1d_1+\beta_2d_2}} &\geq \left(C_4L^{-\vartheta}\right)^{\kappa n_0\,L^{\beta_1d_1+\beta_2d_2}}\\ &= \exp\left[C_5\left(\log E^{\vartheta/2}+\log C_6\right)E^{-(\beta_1d_1+\beta_2d_2)/2}\right].\end{aligned} \tag{86}$$

Here the first inequality derives from Assumption 2.1 on the probability measure of $\mu(\Lambda_0)$. Moreover, the existence of two constants $0<C_5,\,C_6<\infty$ ensuring the validity of the equality follows from (83). Since the choice (83) of the energy-dependence of L guarantees that the pre-factor in the lower bound in Proposition 3.2 is negligible, the proof of Theorem 2.8 is completed by inserting (86) in the left-hand side of (23). $\square$

Appendix A. Proof of mixing of random Borel measure

The purpose of this short appendix is to proof Lemma 2.3. We let $\Lambda^{(n)} := \bigcup_{|j|\leq n} \Lambda_j$ with $n \in \mathbb{N}$. Moreover, let $\mathcal{M}\big(\Lambda^{(n)}\big) \subset \mathcal{M}\big(\mathbb{R}^d\big)$ denote the set of Borel measures with support in $\Lambda^{(n)}$ and let $\mathcal{B}(\mathcal{M}_n)$ be the smallest σ-algebra, which renders the mappings $\mathcal{M}\big(\Lambda^{(n)}\big) \ni \nu \mapsto \nu(\Lambda)$ measurable for all Borel sets $\Lambda \subset \Lambda^{(n)}$. Their union $\mathcal{R} := \bigcup_{n\in\mathbb{N}} \mathcal{B}(\mathcal{M}_n)$ satisfies:

(i) $\mathcal{R}$ generates the σ-algebra $\mathcal{B}(\mathcal{M})$.

(ii) $\mathcal{R}$ is a semiring.

The first assertion holds by definition of $\mathcal{B}(\mathcal{M})$. To check the second one we note that $\emptyset \in \mathcal{R}$. Moreover, for every $M,\, M' \in \mathcal{R}$ there exists some $n \in \mathbb{N}$ such that

$$M,\, M' \in \mathcal{B}(\mathcal{M}_n) \tag{87}$$

and hence $M \cap M' \in \mathcal{B}(\mathcal{M}_n) \subset \mathcal{R}$ and $M \backslash M' \in \mathcal{B}(\mathcal{M}_n) \subset \mathcal{R}$.

Our next aim is to prove the claimed limit relation (11) for all $M,\, M' \in \mathcal{B}(\mathcal{M}_n)$ with $n \in \mathbb{N}$ arbitrary. Assumption 2.1(ii) ensures that the events $T_j M \subset \mathcal{M}\big(\Lambda^{(n)} + j\big)$ and $M' \subset \mathcal{M}\big(\Lambda^{(n)}\big)$ are stochastically independent for all $j \in \mathbb{Z}^d$ with $\big(\Lambda^{(n)} + j\big) \cap \Lambda^{(n)} = \emptyset$, such that

$$\mathcal{P}\left\{T_j M \cap M'\right\} = \mathcal{P}\left\{T_j M\right\} \mathcal{P}\left\{M'\right\} = \mathcal{P}\left\{M\right\} \mathcal{P}\left\{M'\right\}. \tag{88}$$

Here the last equality is a consequence of Assumption 2.1(i).

Thanks to (87) we have thus proven the validity of (88) for all $M,\, M' \in \mathcal{R}$. Lemma 2.3 now follows from [**DVJ88**, Lemma 10.3.II], which is a monotone-class argument. □

Remark A.1. We proved above that the random potential V_ω is mixing under our assumptions. Note, that mixing is actually a property of the probability measure $\mathcal{P}$ with respect to the shifts $\{T_j\}$. However, the potential V_ω will not satify stronger mixing condition such as $\phi-$mixing. In fact, as a rule, the potential may even be deterministic (in the technical sense of this notion, see e.g. [**KKS85**]), which allows mixing, but not $\phi-$mixing. For further references to this see [**Bil68, KM83a**].

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E-mail address: werner.kirsch@mathphys.ruhr-uni-bochum.de

INSTITUT FÜR MATHEMATIK, RUHR-UNIVERSITÄT BOCHUM UND SFB TR 12, 44780 BOCHUM, GERMANY

E-mail address: swarzel@princeton.edu

Present address: PRINCETON UNIVERSITY, DEPTARTMENT OF PHYSICS, JADWIN HALL, PRINCETON, NJ 08544, USA

On leave from:, INSTITUT FÜR THEORETISCHE PHYSIK, UNIVERSITÄT ERLANGEN-NÜRNBERG, STAUDT-STR. 7, 91058 ERLANGEN, GERMANY